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Rational Moment Problems for Compact Sets

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The following "rational" moment problem is discussed. Given distinct real numbers $\lambda_1, \lambda_2, \ldots, \lambda_p$ (the "poles" of the problem), real numbers c_0 and $c_j^{(i)}$ ($j = 1, 2, 3, \ldots; i = 1, 2, \ldots, p$), and a non-empty compact subset K of $(-\infty, +\infty)$, find necessary and sufficient conditions that there exist a non-negative Borel measure μ, supported on K, such that $c_0 = \int_K d\mu(t)$ and $c_j^{(i)} = \int_K (t - \lambda_i)^{-j}\, d\mu(t)$ for $j = 1, 2, 3, \ldots$ and $i = 1, 2, \ldots, p$.

1. INTRODUCTION

In this paper we consider the following "rational" moment problem. Let $\lambda_1, \lambda_2, \ldots, \lambda_p$ be distinct real numbers, let $\{c_j^{(i)}\}_{j=1}^{\infty}$, $i = 1, 2, \ldots, p$, be sequences of real numbers, let c_0 be a real number, and let K be a non-empty compact subset of $\mathbb{R} = (-\infty, +\infty)$. Find necessary and sufficient conditions that there exist a non-negative Borel measure μ, supported on K, such that

$$c_0 = \int_K d\mu(t),$$
$$c_j^{(i)} = \int_K \frac{d\mu(t)}{(t - \lambda_i)^j}, \qquad j = 1, 2, 3, \ldots; i = 1, 2, \ldots, p. \tag{1}$$

The points $\lambda_1, \lambda_2, \ldots, \lambda_p$ will be called the *poles* of the problem. We also consider rational moment problems with a pole at ∞, and rational moment problems having a countable number of poles.

Rational moment problems are studied in [7–9, 11–15] in cases in which the domain of integration K is an interval. In these papers many of the results of the classical moment problems of Stieltjes and Hamburger [1] are extended to various types of rational moment problems, and a theory of orthogonal and quasi-orthogonal rational functions with specified poles

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is developed, analogous to the theory of orthogonal and quasi-orthogonal polynomials [1, 3, 5].

In this paper we establish solvability criteria for rational moment problems with arbitrary poles $\lambda_1, \lambda_2, \ldots, \lambda_p$ (or $\lambda_1, \lambda_2, \lambda_3, \ldots$ in the case of a countable number of poles) and arbitrary compact supporting set K. The criteria are that certain linear functionals determined by the geometry of the compact set K be non-negative when applied to a certain class of rational functions whose poles are those of the given moment problem. These criteria are essentially that certain quadratic forms be non-negative on the vector space spanned by the given basis functions 1, $(t-\lambda_1)^{-1}$, $(t-\lambda_1)^{-2}, \ldots, (t-\lambda_2)^{-1}, (t-\lambda_2)^{-2}, \ldots$. Analogous results for the case in which K is an interval are given in [8, 9, 11, 14]. Our solvability criteria for rational moment problems will be derived from a variant of M. Riesz's theorem on the solvability of polynomial moment problems [1, p. 71; 17].

It will be useful to consider the truncated moment problem in which Eq. (1) is replaced by

$$\begin{aligned} c_0 &= \int_K d\mu(t), \\ c_j^{(i)} &= \int_K \frac{d\mu(t)}{(t-\lambda_i)^j}, \qquad j = 1, 2, \ldots, 2n_i;\ i = 1, 2, \ldots, p, \end{aligned} \tag{2}$$

where $2n_1, 2n_2, \ldots, 2n_p$ are positive even integers. For all positive integers $k_1, k_2, \ldots, k_p$, and k, let $\mathscr{P}(k)$ be the set of all polynomials of degree $\leq k$ with real coefficients, and let $\mathscr{R}(k_1, k_2, \ldots, k_p)$ be the set of all rational functions R of the form

$$R(t) \equiv \alpha_0 + \sum_{i=1}^{p} \sum_{j=1}^{k_i} \frac{\alpha_{ij}}{(t-\lambda_i)^j}, \tag{3}$$

where the coefficients $\alpha_0, \alpha_{ij} \in \mathbb{R}$. By the partial fractions decomposition, $\mathscr{R}(k_1, k_2, \ldots, k_p)$ is the set of all rational functions R of the form

$$R(t) \equiv \frac{P(t)}{(t-\lambda_1)^{k_1}(t-\lambda_2)^{k_2} \cdots (t-\lambda_p)^{k_p}}$$

with $P \in \mathscr{P}(k_1 + k_2 + \cdots + k_p)$. Define the linear functional $\varphi = \varphi_{(k_1, k_2, \ldots, k_p)}$ on $\mathscr{R}(k_1, k_2, \ldots, k_p)$ by setting

$$\varphi(R) = \alpha_0 c_0 + \sum_{i=1}^{p} \sum_{j=1}^{k_i} \alpha_{ij} c_j^{(i)}$$

whenever (3) holds. Then μ is a solution of the truncated moment problem (2) if and only if

$$\varphi_{(2n_1, 2n_2, \dots, 2n_p)}(R) = \int_K R(t)\, d\mu(t)$$

whenever $R \in \mathscr{R}(2n_1, 2n_2, \dots, 2n_p)$.

In order to relate the truncated rational moment problem (2) to M. Riesz's theorem for the truncated polynomial moment problem, for each p-tuple of positive integers $(k_1, k_2, \dots, k_p)$, let $k = k_1 + k_2 + \cdots + k_p$ and define the linear functional $\mathscr{S} = \mathscr{S}_{(k_1, k_2, \dots, k_p)}$ on $\mathscr{P}(k)$ by first setting

$$s_j = s_j(k_1, k_2, \dots, k_p) = \varphi\left(\frac{t^j}{(t - \lambda_1)^{k_1} \cdots (t - \lambda_p)^{k_p}}\right), \quad j = 0, 1, \dots, k$$

and then setting

$$\mathscr{S}(P) = \sum_{j=0}^{k} \xi_j s_j$$

whenever $P(t) \equiv \sum_{j=0}^{k} \xi_j t^j \in \mathscr{P}(k)$.

Note that the definition of the functional $\varphi_{(k_1, k_2, \dots, k_p)}$ is essentially independent of the choice of $k_1, k_2, \dots, k_p$, but that the definition of the functional $\mathscr{S}_{(k_1, k_2, \dots, k_p)}$ is not: $\varphi_{(k_1, k_2, \dots, k_p)}$ is always the restriction of $\varphi_{(k_1 + l_1,\ k_2 + l_2, \dots, k_p + l_p)}$ to the smaller domain $\mathscr{R}(k_1, k_2, \dots, k_p)$, but $\mathscr{S}_{(k_1, k_2, \dots, k_p)}$ is not, in general, the restriction of $\mathscr{S}_{(k_1 + l_1,\ k_2 + l_2, \dots, k_p + l_p)}$ to the smaller domain $\mathscr{P}(k_1 + k_2 + \cdots + k_p)$. For example, in the case of a single pole ($p = 1$) at 0 ($\lambda_1 = 0$), the functionals $\mathscr{S}_{(1)}, \mathscr{S}_{(2)}, \mathscr{S}_{(3)}, \dots$ are determined by the sequences

$$\begin{array}{llll} s_0 = c_1^{(1)}, & s_1 = c_0 & & \\ s_0 = c_2^{(1)}, & s_1 = c_1^{(1)}, & s_2 = c_0 & \\ s_0 = c_3^{(1)}, & s_1 = c_2^{(1)}, & s_2 = c_1^{(1)}, & s_3 = c_0 \\ \dots, & & & \end{array}$$

respectively.

For each choice of p positive even integers $2n_1, 2n_2, \dots, 2n_p$, there is a truncated polynomial moment problem associated with the truncated rational moment problem (2): find necessary and sufficient conditions that there exist a non-negative Borel measure ν, supported on K, such that

$$s_j(2n_1, 2n_2, \ldots, 2n_p) = \int_K t^j \, d\nu(t), \qquad j = 0, 1, \ldots, 2n,$$

where $2n = 2n_1 + 2n_2 + \cdots + 2n_p$. A solution ν of this truncated polynomial moment problem yields a solution μ of the truncated rational moment problem (2), namely

$$d\mu(t) = (t - \lambda_1)^{2n_1}(t - \lambda_2)^{2n_2} \cdots (t - \lambda_p)^{2n_p} \, d\nu(t).$$

That the non-negative Borel measure μ is in fact a solution of the truncated rational moment problem (2) follows from the observation that the functions

$$\frac{t^j}{(t - \lambda_1)^{2n_1}(t - \lambda_2)^{2n_2} \cdots (t - \lambda_p)^{2n_p}}, \qquad j = 0, 1, \ldots, 2n$$

span $\mathscr{R}(2n_1, 2n_2, \ldots, 2n_p)$, and from the equations

$$\begin{aligned}
\varphi&\left(\frac{t^j}{(t - \lambda_1)^{2n_1}(t - \lambda_2)^{2n_2} \cdots (t - \lambda_p)^{2n_p}} \right) \\
&= s_j(2n_1, 2n_2, \ldots, 2n_p) \\
&= \int_K t^j \, d\nu(t) \\
&= \int_K \frac{t^j}{(t - \lambda_1)^{2n_1}(t - \lambda_2)^{2n_2} \cdots (t - \lambda_p)^{2n_p}} \, d\mu(t), \\
&\qquad\qquad\qquad\qquad j = 0, 1, \ldots, 2n.
\end{aligned}$$

2. Results for Polynomial Moment Problems

Associated with every sequence of real numbers $\{s_j\}_{j=0}^{k}$ is a linear functional $\mathscr{S}$ defined on the set of polynomials $P(t) \equiv \sum_{j=0}^{k} \xi_j t^j$ of degree $\leq k$ by

$$\mathscr{S}(P) = \sum_{j=0}^{k} \xi_j s_j.$$

The functional $\mathscr{S}$, and sequence $\{s_j\}_{j=0}^{k}$, are said to be *non-negative on* K, a subset of $\mathbb{R}$, if and only if $\mathscr{S}(P) \geq 0$ for every polynomial P of degree $\leq k$ which is non-negative everywhere on K. The relationship between the non-negativity of sequences and the solvability of polynomial moment problems was discovered by M. Riesz [17]. The following variant of his well-known theorem is the basis for our results (see [2] for related results).

THEOREM 1. *Let K be a non-empty compact subset of $\mathbb{R}$, and let $2n$ be a positive even integer. Then there exists a non-negative Borel measure ν, supported on K, such that*

$$s_j = \int_K t^j \, d\nu(t), \qquad j = 0, 1, \ldots, 2n \tag{4}$$

if and only if $\{s_j\}_{j=0}^{2n}$ is non-negative on K.

Proof. The condition that $\{s_j\}_{j=0}^{2n}$ be non-negative on K is clearly necessary. To prove that it is sufficient, we use the following theorem on the extension of non-negative linear functionals [1, Theorem 2.6.2, p. 69].

Suppose that $\mathfrak{E}$ is a real vector space, that $\mathfrak{M}$ is a vector subspace of $\mathfrak{E}$, that $\mathfrak{K}$ is a convex cone in $\mathfrak{E}$ (i.e., $\mathfrak{K}$ is a convex subset of $\mathfrak{E}$ such that $\alpha f \in \mathfrak{K}$ whenever $\alpha \geq 0$ and $f \in \mathfrak{K}$), and that $\mathscr{S}$ is a real-linear functional on $\mathfrak{M}$ which is non-negative on $\mathfrak{K} \cap \mathfrak{M}$. Then $\mathscr{S}$ can be extended to a real-linear functional $\mathscr{S}'$ on $\mathfrak{E}$ which is non-negative on $\mathfrak{K}$, provided that for every $f \in \mathfrak{E}$ there exist $m_1, m_2 \in \mathfrak{M}$ such that $(m_1 - f) \in \mathfrak{K}$ and $(f - m_2) \in \mathfrak{K}$.

Let $\mathfrak{E} = C(K)$, the set of all real-valued continuous functions with domain K; let $\mathfrak{M}$ be the set of all restrictions to K of polynomials belonging to $\mathscr{P}(2n)$; and let $\mathfrak{K}$ be the set of all functions belonging to $\mathfrak{E} = C(K)$ which are non-negative everywhere on K. Define the real-linear functional $\mathscr{S}$ on $\mathfrak{M}$ by

$$\mathscr{S}(P) = \sum_{j=0}^{2n} \xi_j s_j$$

whenever

$$P(t) = \sum_{j=0}^{2n} \xi_j t^j, \qquad t \in K.$$

By the hypothesis that $\{s_j\}_{j=0}^{2n}$ is non-negative on K, $\mathscr{S}$ is non-negative on $\mathfrak{K} \cap \mathfrak{M}$. The constant function $h(t) = 1$, $t \in K$, belongs to $\mathfrak{M}$, and for every function $g \in \mathfrak{E} = C(K)$ the functions $(\|g\|h - g)$ and $(g + \|g\|h)$

are non-negative everywhere on K and hence belong to $\mathfrak{K}$. (Here $\|g\| = \max\{|g(t)|: t \in K\}$, $g \in C(K)$.) By the theorem cited above, there exists an extension of $\mathscr{S}$ to a real-linear function $\mathscr{S}'$ on $\mathfrak{C}$ which is non-negative on $\mathfrak{K}$.

Since $\mathscr{S}'$ is non-negative on $\mathfrak{K}$, and since $(\|g\|h - g), (g + \|g\|h) \in \mathfrak{K}$ for every $g \in C(K)$, it follows that

$$\mathscr{S}'(g) \le \|g\| \cdot \mathscr{S}'(h) = \|g\| \cdot \mathscr{S}(h) = \|g\| \cdot s_0,$$
$$\mathscr{S}'(g) \ge -\|g\| \cdot \mathscr{S}'(h) = -\|g\| \cdot \mathscr{S}(h) = -\|g\| \cdot s_0,$$

and hence that

$$|\mathscr{S}'(g)| \le s_0 \cdot \|g\|, \qquad g \in C(K).$$

By the Riesz representation theorem [4, p. 265], there exists a non-negative Borel measure ν on K such that

$$s_j = \mathscr{S}(t^j) = \mathscr{S}'(t^j) = \int_K t^j \, d\nu(t)$$

for $t = 0, 1, \ldots, 2n$. ∎

The following variant of a lemma of Kreĭn and Nudel'man [10] provides a characterization of the polynomials which are positive everywhere on a compact set K obtained from a bounded closed interval $[a, b]$ by removing a *finite* number of disjoint open subintervals (see also [6]).

LEMMA. *Suppose that* $-\infty < a \le \alpha_1 < \beta_1 \le \alpha_2 < \beta_2 \le \cdots \le \alpha_M < \beta_M \le b < +\infty$, *and that* P *is a polynomial which is positive everywhere on*

$$K = [a, b] \setminus \bigcup_{m=1}^{M} (\alpha_m, \beta_m).$$

Then

$$P(t) \equiv \sum_{J \subseteq \mathscr{M}} \prod_{m \in J} (t - \alpha_m)(t - \beta_m) P_J(t),$$

where $\mathscr{M} = \{1, 2, \ldots, M\}$ *and, for every* $J \subseteq \mathscr{M}$, P_J *is a polynomial which is non-negative everywhere on* $[a, b]$ *with* $\deg(P_J) + 2 \operatorname{card}(J) \le \deg(P)$.

Note. This lemma is proved by Kreĭn and Nudel'man [10, pp. 292–293, 307] in the case where

$$-\infty < a < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_M < \beta_M < b < +\infty;$$

we need to establish the result in the case in which K may contain one or more isolated points.

Proof. Since P is positive at every point of each of the closed subintervals

$$[a,\alpha_1],[\beta_1,\alpha_2],\ldots,[\beta_{M-1},\alpha_M],[\beta_M,b]$$

(some of which may consist of a single point), the number of zeros of P in each of the open intervals

$$(\alpha_1,\beta_1),\ldots,(\alpha_M,\beta_M)$$

must be even, provided they are counted according to multiplicity. Let $t_1,t_2,\ldots,t_{2l}$ be these zeros. Then

$$P(t)\equiv Q(t)\prod_{j=1}^{2l}(t-t_j),$$

where Q is a polynomial with no zeros in $[a,b]$. Since both $P(t)$ and $\prod_{j=1}^{2l}(t-t_j)$ are positive at the endpoints $t=a$ and $t=b$, Q is also positive at both a and b, and hence on $[a,b]$.

Suppose that $t_1,t_2,\ldots,t_{2k}$ are the zeros of P in (α_1,β_1). Then each $t-t_j$ is a convex combination of $t-\alpha_1$ and $t-\beta_1$,

$$t-t_j=A_j(t-\alpha_1)+B_j(t-\beta_1),$$

where

$$A_j=\frac{\beta_1-t_j}{\beta_1-\alpha_1}>0,\qquad B_j=\frac{t_j-\alpha_1}{\beta_1-\alpha_1}>0.$$

Hence

$$\begin{aligned}\prod_{j=1}^{2k}(t-t_j)&=\prod_{j=1}^{2k}\left[A_j\cdot(t-\alpha_1)+B_j\cdot(t-\beta_1)\right]\\&=Q_1(t)+Q_2(t)\cdot(t-\alpha_1)(t-\beta_1),\end{aligned}$$

where Q_1 and Q_2 are sums of polynomials of the form

$$C\cdot(t-\alpha_1)^{2r}(t-\beta_1)^{2s},\qquad C>0;\ r,s=0,1,2,\ldots$$

such that $\deg(Q_1)\le 2k$, $\deg(Q_2)\le 2k-2$.

Applying a similar argument to each of the remaining open intervals $(\alpha_2,\beta_2),\ldots,(\alpha_M,\beta_M)$ and combining the results, we have the desired conclusion. ∎

3. Solvability Criteria for Rational Moment Problems

Let K be a non-empty compact set with $a = \min K$, $b = \max K$. Then the set of bounded components of the complement of K in $\mathbb{R}$ is an at most countable (possibly empty) collection $\{(\alpha_m, \beta_m): m \in \Omega\}$ of pairwise disjoint, bounded open intervals, each contained in $[a, b]$, such that

$$K = [a, b] \setminus \bigcup_{m \in \Omega} (\alpha_m, \beta_m).$$

Proposition 1. *There exists a solution μ of the truncated rational moment problem* (2) *if and only if*

$$\mathscr{S}_{(2n_1, 2n_2, \ldots, 2n_p)}\Big(\prod_{m \in J} (t - \alpha_m)(t - \beta_m) Q(t)\Big) \geq 0 \tag{5}$$

whenever J is a finite subset of Ω and Q is a polynomial non-negative everywhere on $[a, b]$ with degree $\leq 2n_1 + 2n_2 + \cdots + 2n_p - 2$ card(J). (The product $\prod_{m \in J}(t - \alpha_m)(t - \beta_m)$ is taken to be 1 when J is the empty set.)

Proof. Suppose there is a solution μ. Then, with $\mathscr{S} = \mathscr{S}_{(2n_1, 2n_2, \ldots, 2n_p)}$, we have that

$$\mathscr{S}\Big(\prod_{m \in J} (t - \alpha_m)(t - \beta_m) Q(t)\Big)$$
$$= \int_K \prod_{m \in J} (t - \alpha_m)(t - \beta_m) \cdot \frac{Q(t)}{(t - \lambda_1)^{2n_1} \cdots (t - \lambda_p)^{2n_p}}\, d\mu(t) \geq 0,$$

since $\prod_{m \in J}(t - \alpha_m)(t - \beta_m) \geq 0$ whenever $t \in K$, with J and Q as above.

Conversely, suppose that (5) holds for all such J and Q. Let P be any polynomial which has degree $\leq 2n$, where $n = n_1 + n_2 + \cdots + n_p$, and which is non-negative everywhere on K. Let $\varepsilon > 0$ and set $P_\varepsilon(t) \equiv P(t) + \varepsilon$. Then P_ε has degree $\leq 2n$ and is positive on K. Let $\mathscr{M}$ be the set of all indices $m \in \Omega$ for which P_ε is zero or negative at some point of (α_m, β_m). For each $m \in \Omega$, P_ε is positive at both the endpoints α_m and β_m of (α_m, β_m), and $\alpha_m < \beta_m$; hence if P_ε is negative at some point of (α_m, β_m) then P_ε has a zero in (α_m, β_m). Thus $\mathscr{M}$ is the set of all indices $m \in \Omega$ such that P_ε has a zero in (α_m, β_m), and therefore $\mathscr{M}$ is finite. By the lemma, applied to P_ε on the set $[a, b] \setminus \bigcup_{m \in \mathscr{M}}(\alpha_m, \beta_m)$, we have that

$$P_\varepsilon(t) \equiv \sum_{J \subseteq \mathscr{M}} \prod_{m \in J} (t - \alpha_m)(t - \beta_m) Q_J(t),$$

where each Q_J is a polynomial which is non-negative everywhere on $[a, b]$ with $\deg(Q_J) + 2\operatorname{card}(J) \le \deg(P_\varepsilon) \le 2n$. By (5) and the linearity of $\mathscr{S}$,

$$\mathscr{S}(P_\varepsilon) = \sum_{J \subseteq \mathscr{M}} \mathscr{S}\left(\prod_{m \in J} (t - \alpha_m)(t - \beta_m) Q_J(t) \right) \ge 0.$$

Since $\mathscr{S}$ is linear, it follows that $\mathscr{S}(P) = \mathscr{S}(P_\varepsilon - \varepsilon \cdot 1) = \mathscr{S}(P_\varepsilon) - \varepsilon \mathscr{S}(1) = \mathscr{S}(P_\varepsilon) - \varepsilon s_0 \ge -\varepsilon s_0$ for every $\varepsilon > 0$. Therefore

$$\mathscr{S}(P) \ge 0.$$

Since this inequality holds for every polynomial P which is non-negative everywhere on K and has degree $\le 2n$, M. Riesz's theorem implies that there exists a non-negative Borel measure ν, supported on K, such that

$$s_j = \int_K t^j \, d\nu(t), \qquad j = 0, 1, \ldots, 2n.$$

Let

$$d\mu(t) = (t - \lambda_1)^{2n_1} \cdots (t - \lambda_p)^{2n_p} \, d\nu(t).$$

Then μ is a non-negative Borel measure, supported on K, such that

$$\varphi\left(\frac{t^j}{(t - \lambda_1)^{2n_1} \cdots (t - \lambda_p)^{2n_p}} \right) = \int_K \frac{t^j}{(t - \lambda_1)^{2n_1} \cdots (t - \lambda_p)^{2n_p}} \, d\mu(t),$$
$$j = 0, 1, \ldots, 2n$$

and hence such that (2) holds. ∎

PROPOSITION 2. *There exists a solution μ of the truncated rational moment problem* (2) *if and only if*

$$\mathscr{S}_{(2n_1, 2n_2, \ldots, 2n_p)}\left(\prod_{m \in J} (t - \alpha_m)(t - \beta_m) A(t)^2 \right) \ge 0,$$

$$\mathscr{S}_{(2n_1, 2n_2, \ldots, 2n_p)}\left((t - a) \prod_{m \in J} (t - \alpha_m)(t - \beta_m) B(t)^2 \right) \ge 0,$$

$$\mathscr{S}_{(2n_1, 2n_2, \ldots, 2n_p)}\left((b - t) \prod_{m \in J} (t - \alpha_m)(t - \beta_m) C(t)^2 \right) \ge 0,$$

$$\mathscr{S}_{(2n_1, 2n_2, \ldots, 2n_p)}\left((b - t)(t - a) \prod_{m \in J} (t - \alpha_m)(t - \beta_m) D(t)^2 \right) \ge 0,$$

whenever J is a finite subset of Ω and A, B, C, D are polynomials with real *coefficients whose degrees are small enough that the arguments of $\mathscr{S}$ in the inequalities above have degree $\leq 2n$.*

Proof. By a theorem of Lukács [16, Prob. 47, pp. 78, 260], a polynomial P is non-negative at every point of $[a, b]$ if and only if

$$P(t) \equiv A(t)^2 + (t-a)B(t)^2 + (b-t)C(t)^2 + (b-t)(t-a)D(t)^2,$$

where A, B, C, D are polynomials with real coefficients such that each of the terms in the sum has degree $\leq \deg(P)$. Hence this proposition follows from Proposition 1 and Lukács' theorem. ▮

Let $\mathscr{R}$ be the set of all rational functions of the form (3) with $k_1, k_2, \ldots, k_p$ arbitrary, and define the linear functional Φ on $\mathscr{R}$ by setting

$$\Phi(R) \equiv \alpha_0 c_0 + \sum_{i=1}^{p} \sum_{j=1}^{k_i} \alpha_{ij} c_j^{(i)}$$

whenever (3) holds.

THEOREM 2. *The full rational moment problem* (1) *has a solution* μ *if and only if*

$$\Phi\left(\frac{\prod_{m \in J} (t-\alpha_m)(t-\beta_m) A(t)^2}{(t-\lambda_1)^{2n_1}(t-\lambda_2)^{2n_2} \cdots (t-\lambda_p)^{2n_p}} \right) \geq 0,$$

$$\Phi\left(\frac{(t-a)\prod_{m \in J} (t-\alpha_m)(t-\beta_m) B(t)^2}{(t-\lambda_1)^{2n_1}(t-\lambda_2)^{2n_2} \cdots (t-\lambda_p)^{2n_p}} \right) \geq 0,$$

$$\Phi\left(\frac{(b-t)\prod_{m \in J} (t-\alpha_m)(t-\beta_m) C(t)^2}{(t-\lambda_1)^{2n_1}(t-\lambda_2)^{2n_2} \cdots (t-\lambda_p)^{2n_p}} \right) \geq 0,$$

$$\Phi\left(\frac{(b-t)(t-a)\prod_{m \in J} (t-\alpha_m)(t-\beta_m) D(t)^2}{(t-\lambda_1)^{2n_1}(t-\lambda_2)^{2n_2} \cdots (t-\lambda_p)^{2n_p}} \right) \geq 0,$$

whenever J is a finite subset of Ω; $n_1, n_2, \ldots, n_p$ are positive integers, and A, B, C, D are polynomials with real coefficients whose degrees are small enough that the arguments of Φ in the inequalities above belong to $\mathscr{R}$.

Proof. It is clear that the inequalities above hold if there is a solution μ of the full rational moment problem. Conversely, suppose that the inequalities above hold. By Proposition 2, for every p-tuple $N = (2n_1, 2n_2, \ldots, 2n_p)$ of positive even integers there exists a non-negative Borel measure μ_N, supported on K, which is a solution of the truncated rational moment problem (2). Applying Helly's theorems [3, pp. 53–54; 5, p. 56] to the family of measures $\{\mu_N\}$, we obtain a solution μ of the full rational moment problem, as in [8, pp. 551–553]. ▮

COROLLARY. *The full rational moment problem* (1) *has a solution* μ *if and only if*

$$\Phi\left(\frac{P(t)}{(t-\lambda_1)^{2n_1}(t-\lambda_2)^{2n_2}\cdots(t-\lambda_p)^{2n_p}}\right) \geq 0 \tag{6}$$

whenever $n_1, n_2, \ldots, n_p$ *are positive integers and* P *is a polynomial with real coefficients with degree* $\leq 2n_1 + 2n_2 + \cdots + 2n_p$, *which is non-negative everywhere on* K.

We should like to assert that there exists a solution μ of (1) if and only if $\Phi(R) \geq 0$ for every $R \in \mathscr{R}$ which is non-negative everywhere on K. If none of the poles $\lambda_1, \lambda_2, \ldots, \lambda_p$ belongs to K then this assertion follows from the Corollary. But if one of the poles, say λ_i, belongs to K, then it is difficult to assign a meaning to the statement that R is non-negative everywhere on K if R is a rational function with a pole at λ_i; in this case the Corollary is the closest we can come to such an assertion. Note that, in all cases, the set of rational functions

$$\frac{P(t)}{(t-\lambda_1)^{2n_1}(t-\lambda_2)^{2n_2}\cdots(t-\lambda_p)^{2n_p}}$$

of the form specified in the Corollary is a positive cone in $\mathscr{R}$.

4. Other Rational Moment Problems; Uniqueness

We consider the extension of the rational moment problem (1) to one having a countable number of real poles: let $\lambda_1, \lambda_2, \lambda_3, \ldots,$ be distinct real numbers, let $\{c_j^{(i)}\}_{j=1}^{\infty}$, $i = 1, 2, 3, \ldots,$ be sequences of real numbers, let c_0 be a real number, and let K be a non-empty compact subset of $\mathbb{R}$; find necessary and sufficient conditions that there exist a non-negative

Borel measure μ, supported on K, such that

$$c_0 = \int_K d\mu(t),$$
$$c_j^{(i)} = \int_K \frac{d\mu(t)}{(t-\lambda_i)^j}, \qquad i, j = 1, 2, 3, \ldots . \tag{7}$$

THEOREM 3. *The rational moment problem with countably many poles (7) has a solution μ, supported on K, if and only if the inequalities in Theorem 2 hold for all positive integers p and all p-tuples of positive even integers $(2n_1, 2n_2, \ldots, 2n_p)$.*

Proof. It is clear that the inequalities hold if μ is a solution of (7). Conversely, suppose that these inequalities hold. By Theorem 2, for every positive integer p there is a non-negative Borel measure μ_p, supported on K, such that (1) holds. Applying Helly's theorems to the sequence of measures $\{\mu_p\}_{p=1}^{\infty}$ as in [8, pp. 551–553], we obtain a non-negative Borel measure μ, supported on K, which satisfies (1). ▮

By a rational moment problem with a pole at ∞, we mean a problem of the following type. Let $\lambda_1, \lambda_2, \ldots, \lambda_p$ be distinct real numbers, let $\{c_j^{(0)}\}_{j=0}^{\infty}$ and $\{c_j^{(i)}\}_{j=1}^{\infty}$, $i = 1, \ldots, p$, be sequences of real numbers, and let K be a non-empty compact subset of $\mathbb{R}$; find necessary and sufficient conditions that there exist a non-negative Borel measure μ, supported on K, such that

$$\begin{aligned} c_j^{(0)} &= \int_K t^j \, d\mu(t), \qquad j = 0, 1, 2, \ldots, \\ c_j^{(i)} &= \int_K \frac{d\mu(t)}{(t-\lambda_i)^j}, \qquad j = 1, 2, 3, \ldots; i = 1, \ldots, p. \end{aligned} \tag{8}$$

For such a problem we must extend the domain of the linear functional Φ to the set of rational functions R of the form

$$R(t) \equiv \sum_{j=0}^{k_0} \alpha_{0j} t^j + \sum_{i=1}^{p} \sum_{j=1}^{k_i} \frac{\alpha_{ij}}{(t-\lambda_i)^j} \tag{9}$$

with $\alpha_{0j}, \alpha_{ij} \in \mathbb{R}$, by setting

$$\Phi(R) = \sum_{j=0}^{k_0} \alpha_{0j} c_j^{(0)} + \sum_{i=1}^{p} \sum_{j=1}^{k_i} \alpha_{ij} c_j^{(i)}$$

whenever (9) holds.

THEOREM 4. *The rational moment problem with a pole at* ∞ (8) *has a solution* μ, *supported on* K, *if and only if the inequalities in Theorem* 2 *hold without restriction on the degrees of the polynomials* A, B, C, D.

We omit the proof, which is similar to the proof of Theorem 2.

Finally, we consider rational moment problems with a pole at ∞ and a countable number of real poles: let $\lambda_1, \lambda_2, \lambda_3, \ldots$ be distinct real numbers, let $\{c_j^{(0)}\}_{j=0}^{\infty}$ and $\{c_j^{(i)}\}_{j=1}^{\infty}$, $i = 1, 2, 3, \ldots$, be sequences of real numbers, and let K be a non-empty compact subset of $\mathbb{R}$; find necessary and sufficient conditions that there exist a non-negative Borel measure μ, supported on K, such that

$$c_j^{(0)} = \int_K t^j \, d\mu(t), \qquad j = 0, 1, 2, \ldots,$$

$$c_j^{(i)} = \int_K \frac{d\mu(t)}{(t - \lambda_i)^j}, \qquad j = 1, 2, 3, \ldots; i = 1, 2, 3, \ldots.$$

THEOREM 5. *The rational moment problem with a pole at* ∞ *and countably many real poles* (10) *has a solution* μ, *supported on* K, *if and only if the inequalities in Theorem* 2 *hold for every positive integer* p *and for every* p-*tuple of positive even integers* $(2n_1, 2n_2, \ldots, 2n_p)$, *and without restriction on the degrees of the polynomials* A, B, C, D.

Theorem 5 follows from Theorem 4 by application of Helly's theorems, just as Theorem 3 follows from Theorem 2.

Note that if any of the non-truncated rational moment problems above has a pole λ_i outside K, then the solution of the problem is unique. This follows from the Stone–Weierstrass theorem [4, p. 272] applied to the algebra of real-valued functions generated by 1 and $(t - \lambda_i)^{-1}$, and the Riesz representation theorem. Also, if K is compact and the problem has a pole at ∞, then the solution of the problem is unique, again by the Stone–Weierstrass theorem and the Riesz representation theorem.

5. AN EXAMPLE

Let $\lambda_0, \lambda_1, \lambda_2, \ldots$ be real numbers with $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_0 < 1$ and $\lambda_n \to \lambda_0$ as $n \to \infty$. We construct a *positive* Borel measure μ with finite total mass and $\operatorname{supp}(\mu) = [0, 2]$ such that the functions

$$(t - \lambda_i)^{-j}, t^j, \qquad i = 0, 1, 2 \ldots; j = 0, 1, 2 \ldots$$

belong to $L^1(\mu)$. The existence of such a measure μ demonstrates that none of the rational moment problems considered above is trivial. (Here, a *trivial* problem is one which is solvable only in the special case in which the given moments $c_0, c_j^{(i)}$ are all zero.) In this example, the poles $\lambda_0, \lambda_1, \lambda_2, \ldots$ are all interior points of supp(μ), and hence the example shows that, in the case of a countable number of poles, the poles may have a limit point in supp(μ).

Choose real numbers $\rho_0, \rho_1, \rho_2, \ldots$ such that

$$0 = \rho_0 < \lambda_1 < \rho_1 < \lambda_2 < \rho_2 < \cdots$$

and define the positive Borel measures $\mu_0, \mu_1, \mu_2, \ldots, \mu_\infty$ by

$$d\mu_k(t) = \frac{1}{k^2} \cdot 1_{(\rho_{k-1}, \rho_k)}(t) \cdot \frac{e^{-1/|t-\lambda_k|}}{(t-\lambda_k)^2}\, dt, \qquad k = 1,2,3,\ldots,$$

$$d\mu_0(t) = 1_{(\lambda_0, 1)}(t) \cdot \frac{e^{-1/(t-\lambda_0)}}{(t-\lambda_0)^2}\, dt,$$

$$d\mu_\infty(t) = 1_{(1,2)}(t) \cdot e^{-t}\, dt,$$

where 1_E is the indicator function of E, $E \subseteq \mathbb{R}$. Let

$$\mu = \sum_{k=1}^{\infty} \mu_k + \mu_0 + \mu_\infty.$$

Then μ is a positive Borel measure and supp(μ) = [0, 2].

For each $k = 1,2,3,\ldots$, the change of variable $x = 1/(t - \lambda_k)$ gives

$$\begin{aligned}\mu_k(\mathbb{R}) &= \frac{1}{k^2}\int_{\rho_{k-1}}^{\rho_k} \frac{e^{-1/|t-\lambda_k|}}{(t-\lambda_k)^2}\, dt \\ &= \frac{1}{k^2} \cdot \left(\int_{-\infty}^{-1/(\lambda_k - \rho_{k-1})} + \int_{1/(\rho_k - \lambda_k)}^{+\infty}\right) e^{-|x|}\, dx \\ &\le \frac{1}{k^2} \cdot \int_{-\infty}^{+\infty} e^{-|x|}\, dx = \frac{2}{k^2};\end{aligned}$$

and for $k = 0$ the change of variable $x = 1/(t - \lambda_0)$ gives

$$\mu_0(\mathbb{R}) = \int_{\lambda_0}^{1} \frac{e^{-1/(t-\lambda_0)}}{(t-\lambda_0)^2}\, dt = \int_{1/(1-\lambda_0)}^{+\infty} e^{-x}\, dx \le 1.$$

Also

$$\mu_\infty(\mathbb{R}) = \int_1^2 e^{-t}\,dt \le 1.$$

It follows that

$$\mu(\mathbb{R}) = \sum_{k=1}^{\infty} \frac{2}{k^2} + 2 < +\infty$$

and hence that μ has finite total mass.

The following estimates show that $(t-\lambda_i)^{-j}$, $t^j \in L^1(\mu)$ for $i, j = 0, 1, 2, \ldots$. For $i, j, k = 1, 2, 3, \ldots$ we have that

$$i < k \Rightarrow \int \left|(t-\lambda_i)^{-j}\right| d\mu_k(t) = \int_{\rho_{k-1}}^{\rho_k} \left|(t-\lambda_i)^{-j}\right| d\mu_k(t)$$

$$\le \frac{1}{(\rho_{k-1}-\lambda_i)^j} \cdot \int_{\rho_{k-1}}^{\rho_k} d\mu_k(t) \le \frac{1}{(\rho_{k-1}-\lambda_i)^j} \cdot \frac{2}{k^2}$$

$$\le \frac{1}{(\rho_i-\lambda_i)^j} \cdot \frac{2}{k^2},$$

$$i > k \Rightarrow \int \left|(t-\lambda_i)^{-j}\right| d\mu_k(t) = \int_{\rho_{k-1}}^{\rho_k} \left|(t-\lambda_i)^{-j}\right| d\mu_k(t)$$

$$\le \frac{1}{(\lambda_i-\rho_k)^j} \cdot \int_{\rho_{k-1}}^{\rho_k} d\mu_k(t) \le \frac{1}{(\lambda_i-\rho_k)^j} \cdot \frac{2}{k^2}$$

$$\le \frac{1}{(\lambda_i-\rho_{i-1})^j} \cdot \frac{2}{k^2},$$

$$i = k \Rightarrow \int \left|(t-\lambda_i)^{-j}\right| d\mu_i(t) = \frac{1}{i^2} \cdot \int_{\rho_{i-1}}^{\rho_i} \left|(t-\lambda_i)^{-j}\right| \frac{e^{-1/|t-\lambda_i|}}{(t-\lambda_i)^2}\,dt$$

$$= \frac{1}{i^2} \cdot \left(\int_{-\infty}^{-1/(\lambda_i-\rho_{i-1})} + \int_{1/(\rho_i-\lambda_i)}^{+\infty} \right) |x|^j e^{-|x|}\,dx$$

$$\le \frac{1}{i^2} \cdot \int_{-\infty}^{+\infty} |x|^j e^{-|x|}\,dx = \frac{2j!}{i^2},$$

and also that

$$\int |(t-\lambda_i)^{-j}|\,d\mu_0(t) = \int_{\lambda_0}^{1} (t-\lambda_i)^{-j}\,d\mu_0(t)$$
$$\leq \frac{1}{(\lambda_0-\lambda_i)^j}\mu_0(\mathbb{R}) \leq \frac{1}{(\lambda_0-\lambda_i)^j},$$

$$\int |(t-\lambda_i)^{-j}|\,d\mu_\infty(t) = \int_{1}^{2} (t-\lambda_i)^{-j}\,d\mu_\infty(t)$$
$$\leq \frac{1}{(1-\lambda_i)^j}\mu_\infty(\mathbb{R}) \leq \frac{1}{(1-\lambda_i)^j}.$$

Thus

$$\int |(t-\lambda_i)^{-j}|\,d\mu(t) \leq \frac{1}{(\lambda_i-\rho_{i-1})^j}\cdot\sum_{k=1}^{i-1}\frac{2}{k^2}$$
$$+\frac{1}{(\rho_i-\lambda_i)^j}\cdot\sum_{k=i+1}^{\infty}\frac{2}{k^2}+\frac{2j!}{i^2}$$
$$+\frac{1}{(\lambda_0-\lambda_i)^j}+\frac{1}{(1-\lambda_i)^j}$$
$$< +\infty, \qquad i,j = 1,2,3,\ldots .$$

Similar arguments show that $(t-\lambda_0)^{-j}$, $t^j \in L^1(\mu)$ for $j = 1,2,3,\ldots$.

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